October Math Gems

PROBLEM OF THE WEEK 8

§1 problems

Problem 1.1. Let a, b, c be integers such that $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 3$ prove that abc is a perfect cube using AM-GM inequality!

answer.

$$\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \geq \sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 1$$

Then,

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

Where the equality applies when each ratio is equal to 1. Then, $a = b = c$
Then, $abc = a^3 = b^3 = c^3$ which means that it is perfect cube. □

Problem 1.2. Determine all roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x + y + z &= 3, \\ x^2 + y^2 + z^2 &= 3, \\ x^3 + y^3 + z^3 &= 3. \end{aligned}$$

answer. We begin by manipulating the given equations to find other useful relations between the variables.

$$(\sum x)^2 = \sum x^2 + 2\sum xy$$

$$3^2 = 3 + 2\sum xy$$

$$\sum xy = 3$$

Now using the following identity,

$$\sum x^3 - 3xyz = (\sum x)(\sum x^2 - \sum xy)$$

$$3 - 3xyz = 3(0)$$

$$xyz = 1$$

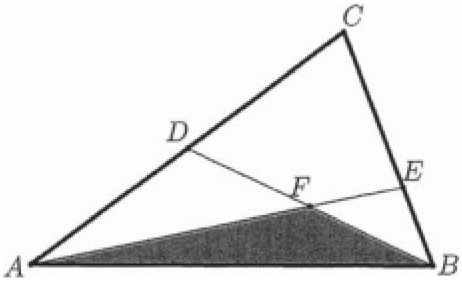
Now, let x, y, and z be the roots of a cubic polynomial $P(t)$. Then, by Vieta's Formulas, we have that

$$P(t) = t^3 - 3t^2 + 3t - 1$$

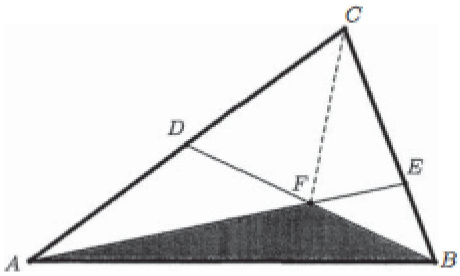
$$P(t) = (t - 1)^3$$

Therefore, $\boxed{x = y = z = 1}$ is the only solution. □

Problem 1.3. Given that ABC is a triangle with D being the midpoint of AC and E a point on CB such that $CE = 2EB$. If AE and BD intersect at point F and the area of triangle AFB is equal to 1 unit, find the area of ABC.



answer. First, we construct a line joining C and F. Let us use the symbol $[XYZ]$ to denote the area of the triangle XYZ. We know that $[ADF] = [DCF] = x$, and if $[BFE] = z$, then $[FCE] = 2z$. Furthermore, we have $[ADB] = [DCB]$, i.e. $x+1 = x+3z$, so $z = \frac{1}{3}$. Also, $2 \times [AEB] = [ACE]$, i.e. $2 + 2z = 2x + 2z$, so $x = 1$. Finally, $[ABC] = 1 + 2x + 3z = 4$ units.



$\square$

Problem 1.4. If $\tan a = \frac{1}{2m+1}$,and $\tan b = \frac{m}{m+m+1}$ where a, b are acute, then $a + b =$

answer. We know that

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{\frac{1}{2m+1} + \frac{m}{m+1}}{1 - \frac{1}{2m+1}\frac{m}{m+1}}$$

$$= \frac{m+1+m(2m+1)}{(1+m)(2m+1)-m} = \frac{2m^2+2m+1}{2m^2+3m+1-m} = 1$$

$$\tan(a+b) = 1 \to a+b = \frac{\pi}{4}$$

$\square$

Problem 1.5. If $f(x) = (a - x^n)^{\frac{1}{n}}$. Prove that $f(x) = f^{-1}(x)$

answer. We already know that $f(f^{-1}(x)) = x$ (i)

$$f(f(x)) = f(a - x^n)^{\frac{1}{n}}$$

$$= (a - ((a - x^n)^{\frac{1}{n}})^n)^{\frac{1}{n}} = (a - (a - x^n))^{\frac{1}{n}} = (a - a + x^n)^{\frac{1}{n}} = x$$

(ii) From (i) and (ii) we get that $f(x) = f^{-1}(x)$ $\square$

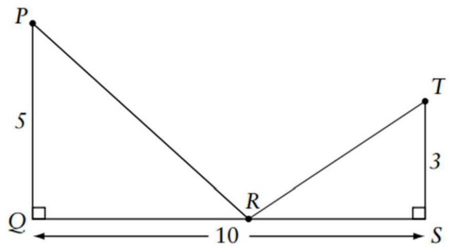
Problem 1.6. Two integers have a sum of 26. When two more integers are added to the first two integers the sum is 41. Finally when two more integers are added to the sum of the previous four integers the sum is 57. What is the minimum number of even integers among the 6 integers?

answer. Suppose the six numbers are a, b, c, d, e, f and

$$\begin{cases} a+b=26 \\ a+b+c+d=41 \\ a+b+c+d+e+f=57 \end{cases} \implies \begin{cases} e+f=16 \\ c+d=15 \\ a+b=26 \end{cases}$$

As, $a+b=26$ so we can say that a and b are both odd or even also, a, b. In $c+d=odd$ so one of them is odd and the other is even. In the question, he wants the minimum possible number of even integer and it will be one. As, we can say that a, b, e, f are odd and one of c and d are even. □

Problem 1.7. A student wishes to move from point P to point T via point R on line QS. What is the distance QR that makes the total distance traveled minimum?



answer. The minimum distance that the student will move is

$$\sqrt{(3+5)^2+10^2}=2\sqrt{41}$$

Suppose that,

$$QR=x \implies RS=10-x$$

Also, suppose that

$$PR=y \implies RT=2\sqrt{41}-y$$

Now, apply Pythagoras theorem in the two triangles

$$\begin{cases} 25+x^2=y^2 \\ 9+(10-x)^2=(2\sqrt{41}-y)^2 \end{cases} \implies x=\frac{25}{4}, \quad y=\frac{5\sqrt{41}}{4}$$

So, the minimum value of QR is $\frac{25}{4}=6.25$. □

Problem 1.8. Simplify the give expression

$$\log_2 3 \times \log_3 4 \times \log_4 5 \times ... \times \log_{127} 128$$

answer. We can use the identity $\log_a b = \frac{ln(b)}{ln(a)}$ to rewrite all the expression and we get

$$\frac{ln(3)}{ln(2)} \times \frac{ln(4)}{ln(3)} \times \frac{ln(5)}{ln(4)} \times ... \times \frac{ln(128)}{ln(127)}$$

We can see that $ln(3)$ cancels, $ln(4)$ cancels and so on. We are, finally, left with

$$\frac{ln(128)}{ln(2)}$$

This can also be written as $\log_2 128$, which is equal to 7. □

Problem 1.9. In $\triangle ABC$ with integer side lengths,

$$\cos A = \frac{11}{16}, \qquad \cos B = \frac{7}{8}, \qquad \text{and} \qquad \cos C = -\frac{1}{4}.$$

What is the least possible perimeter for $\triangle ABC$?

answer. By the law of sines, note that

$$\frac{\sin(A)}{a} = \frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{1}{2R}.$$

This means that the ratio of the side lengths is equal to the ratio of the sines of the corresponding angles. Since we can scale the triangle however we want, we can make R any value, thus scaling the sides by any factor of the sines. We now can compute the sines.

$$\sin(A) = \sqrt{1 - \cos^2(A)} = \tfrac{3\sqrt{15}}{16}$$
$$\sin(B) = \sqrt{1 - \cos^2(B)} = \tfrac{\sqrt{15}}{8}$$
$$\sin(C) = \sqrt{1 - \cos^2(C)} = \tfrac{\sqrt{15}}{4}$$

It's not hard to see that these are in a ratio of $3 : 2 : 4$; and since we can let R be whatever we want it to be, this is also the smallest set of integer sides, leading to a least perimeter of $\boxed{9}$. □

Problem 1.10. be the system

$$\begin{cases} 3x_1^2 + 3x_2^2 + 3x_3^2 = 6x_4 - 1 \\ 3x_1^2 + 3x_2^2 + 3x_4^2 = 6x_3 - 1 \\ 3x_1^2 + 3x_3^2 + 3x_4^2 = 6x_2 - 1 \\ 3x_2^2 + 3x_3^2 + 3x_4^2 = 6x_1 - 1 \end{cases}$$

Calculate the value the

$$\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$$

answer. Adding all the equations results in

$$9(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 6(x_1 + x_2 + x_3 + x_4) - 4$$

or

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = \frac{2}{3}(x_1 + x_2 + x_3 + x_4) - \frac{4}{9}$$

or

$$(x_1 - \frac{1}{3})^2 + (x_2 - \frac{1}{3})^2 + (x_3 - \frac{1}{3})^2 + (x_4 - \frac{1}{3})^2 - \frac{4}{9} = -\frac{4}{9} \Longrightarrow x_1 = x_2 = x_3 = x_4 = \frac{1}{3}$$

$\Longrightarrow$ Sum of inverses=12 □

Problem 1.11. $\frac{1+\sin a-\cos 2a-\sin 3a}{2\sin^2 a+\sin a-1}$ Simpify expression

answer.

$$\frac{1+\sin a-\cos 2a-\sin 3a}{2\sin^2 a+\sin a-1}$$
$$\frac{(1-\cos 2a)+\sin a-\sin(a+2a)}{2\sin^2 a+\sin a-1}$$
$$\frac{2\sin^2 a+\sin a-\sin a\cos 2a-\sin 2a\cos a}{2\sin^2 a+\sin a-1}$$
$$\frac{2\sin^2 a+\sin a-\sin a(1-2\sin^2 a)-2\sin a\cos^2 a}{2\sin^2 a+\sin a-1}$$
$$\frac{2\sin^2 a+\sin a-\sin a(1-2\sin^2 a)-2\sin a(1-\sin^2 a)}{2\sin^2 a+\sin a-1}$$
$$\frac{2\sin a(2\sin^2 a+\sin a-1)}{2\sin^2 a+\sin a-1}$$
$$\boxed{2\sin a}$$

□

Problem 1.12. If

$$\sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right)=\prod_{r=10}^{99}\log_r(r+1)$$

What is the value of n

answer. LHS

$$\sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right)=\log_2\frac{2}{1}+\log_2\frac{2}{3}+\cdots+\log_2\frac{n+1}{n}=$$

$$\log_2\frac{2}{1}\cdot\frac{3}{2}\cdots\frac{n+1}{n}=\log_2(n+1)$$

R.H.S

$$\prod_{r=10}^{99}\log_r(r+1)=\log_{10}11\cdot\log_{11}12\cdots\log_{99}100=$$

$$\frac{\log_2 11}{\log_2 10}\cdot\frac{\log_2 12}{\log_2 11}\cdots\frac{\log_2 100}{\log_2 99}\cdot=\frac{\log_2 100}{\log_2 10}=2$$

Then, $\log_2(n+1)=2\rightarrow n+1=4\rightarrow n=3$ □

Problem 1.13. Known that

$$\frac{x+y}{2}\geq\sqrt{xy}$$

The minimum value of

$$3^{\sin^6(x)}+3^{\cos^6(x)}$$

can be written in the form of ab^c. Find $4ac+b$

answer.

$$\frac{3^{\sin^6 x}+3^{\sin^6 x}}{2} \geq \sqrt{3^{\sin^6 x}.3^{\cos^6 x}}$$
$$\frac{3^{\sin^6 x}+3^{\sin^6 x}}{2} \geq \sqrt{3^{(\sin^6 x+\cos^6 x)}}$$
$$\frac{3^{\sin^6 x}+3^{\sin^6 x}}{2} \geq \sqrt{3^{(1-3\sin^2 x.\cos^2 x)}}$$
$$\frac{3^{\sin^6 x}+3^{\sin^6 x}}{2} \geq \sqrt{3^{(1-\frac{3}{4}\sin^2 2x)}}$$

The expression is minimum only when $\sin^2 2x$ is maximum and equal to 1 ,and the equality holds

$$3^{\sin^6 x}+3^{\sin^6 x}=2\sqrt{3^{\frac{1}{4}}}=2\cdot 3^{\frac{1}{8}}$$

Then, $a=2$, $b=3$,and $c=\frac{1}{8}$ which yields the answer to 4 $\square$

Problem 1.14. Let m, n be positive integers with $m<n$. Find the a closed form for the sum

$$\frac{1}{\sqrt{m}+\sqrt{m+1}}+\frac{1}{\sqrt{m+1}+\sqrt{m+2}}+\cdots+\frac{1}{\sqrt{n-1}+\sqrt{n}}$$

answer. By taking the conjugate for each term of the sum, we get the following:

$$\frac{\sqrt{m+1}-\sqrt{m}}{m+1-m}+\frac{\sqrt{m+2}-\sqrt{m+1}}{m+2-m-1}+\cdots+\frac{\sqrt{n}-\sqrt{n-1}}{n-n+1}$$

which is equal to:

$$\sqrt{m+1}-\sqrt{m}+\sqrt{m+2}-\sqrt{m+1}+\cdots+\sqrt{n}-\sqrt{n-1}=\sqrt{n}-\sqrt{m}$$

$\square$

Problem 1.15. Find all real numbers x for which

$$\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}$$

answer. Let $2^x=a$ and $3^x=b$, then we can rewrite the equation as following:

$$\frac{a^3+b^3}{a^2b+ab^2}=\frac{7}{6}$$

Notice that we can factor out ab from the denominator of the left-hand side.

$$\frac{a^3+b^3}{ab(a+b)}=\frac{7}{6}$$

In order to get the most simpler form, we'll use the following fact: $(a^3+b^3)=(a+b)(a^2-ab+b^2)$

$$\begin{aligned}
&=\frac{(a+b)(a^2-ab+b^2)}{ab(a+b)}=\frac{7}{6}\\
&=\frac{a^2-ab+b^2}{ab}=\frac{7}{6}\\
&=6a^2-13ab+6b^2=0\\
&=(2a-3b)(3a-2b)=0
\end{aligned}$$

Therefore we have two solutions $2^{x+1}=3^{x+1}$ or $2^{x-1}=3^{x-1}$, which implies that x can be equal to 1 or -1. $\square$

Problem 1.16. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right)+12\cos a=5\cos^4\left(\frac{a}{2}\right)+12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a=\frac{m}{n}$. Find $10m+n$.

answer. Rewrite the given equation as

$$12\cos a-12\sin a=5\cos^4\left(\frac{a}{2}\right)-5\sin^4\left(\frac{a}{2}\right)$$

Then,

$$12\cos a-12\sin a=5\left[\cos^2\left(\frac{a}{2}\right)-\sin^2\left(\frac{a}{2}\right)\right]=5\cos a$$

It follows that $\tan a=\frac{\sin a}{\cos a}=\frac{7}{12}$. The requested expression is $10\cdot 7+12=82$. □

Problem 1.17. Let z be a complex number that satisfies the equation

$$\frac{z-4}{z^2-5z+1}+\frac{2z-4}{2z^2-5z+1}+\frac{z-2}{z^2-3z+1}=\frac{3}{z}.$$

Over all possible values of z, find the sum of the values of

$$\left|\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right|.$$

answer. Multiply both sides of the equation by z. We get

$$\frac{z^2-4z}{z^2-5z+1}+\frac{2z^2-4z}{2z^2-5z+1}+\frac{z^2-2z}{z^2-3z+1}=3.$$

We may rewrite the left hand side as

$$\frac{z^2-5z+1+z-1}{z^2-5z+1}+\frac{2z^2-5z+1+z-1}{2z^2-5z+1}+\frac{z^2-3z+1+z-1}{z^2-3z+1}.$$

This becomes

$$1+\frac{z-1}{z^2-5z+1}+1+\frac{z-1}{2z^2-5z+1}+1+\frac{z-1}{z^2-3z+1}=3,$$

or

$$\frac{z-1}{z^2-5z+1}+\frac{z-1}{2z^2-5z+1}+\frac{z-1}{z^2-3z+1}=0.$$

Thus, either $z-1=0$, or the expression in question is equal to 0. If $z-1=0$, then $z=1$, and upon plugging this value into the expression in question gives us

$$\left|\frac{1}{z^2-5z+1}+\frac{1}{2z^2-5z+1}+\frac{1}{z^2-3z+1}\right|=\left|-\frac{1}{3}-\frac{1}{2}-1\right|=\frac{11}{6}.$$

Thus, our answer is $0+\frac{11}{6}=\boxed{\frac{11}{6}}$. □

Problem 1.18. Let ABC be an acute triangle with $\angle ABC=60^\circ$. Suppose points D and E are on lines AB and CB, respectively, such that CDB and AEB are equilateral triangles. Given that the positive difference between the perimeters of CDB and AEB is 60 and $DE=45$, what is the value of $AB\cdot BC$?

answer. let D be outside triangle ABC and $AB = x$. Therefore we are looking for $x(x+20)$. We know that $AD = 20$. Law of Cosines on $\triangle BED$ gives us

$$x^2 + (x+20)^2 - (x)(x+20) = 45^2$$

$$x^2 + 20x = \boxed{1625}$$

□

Problem 1.19. Let $f(x)$ be a quadratic with integer coefficients. Suppose there exist positive primes $p < q$ such that $f(p) = f(q) = 87$ and $f(p+q) = 178$. Find $p^2 + q^2$.

answer. Let $g(x) = f(x) - 87$.
We then have that $g(x) = (x-p)(x-q)$, so $f(x) = (x-p)(x-q) + 87$.
Then, we also have that $178 = f(p+q) = pq + 87$, so we have that $pq = 91$.
This implies that p and q are 7 and 13 in some order, so we have that $p^2 + q^2 = 7^2 + 13^2 = 218$. □

Problem 1.20. Find all pairs (x, y) of real numbers satisfying the system :

$$\begin{cases} x + y = 3 \\ x^4 - y^4 = 8x - y \end{cases}$$

answer. From the first equation, $x + y = 3 \implies y = 3 - x$. Substituting this to the second equation gives

$$x^4 - (3-x)^4 = 8x - (3-x)$$

$$12x^3 - 54x^2 + 99x - 78 = 0$$

$$3(x-2)(4x^2 - 10x + 13) = 0$$

The only real root is $x = 2$.
The only pair is $\boxed{(2, 1)}$. □